Engineering Mathematics

Q1. (a) Solution.

Let f(x) = $\log(1+x)$
f(0) = log 1=0
Then

$$f'(x) = \frac{1}{(1+x)} \quad , \quad f'(0) = 1$$

$$f''(x) = (-1)(1+x)^{-2} \ , \ f''(0) = -1$$

And so on....

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots\ldots\ldots\ldots\ldots\ldots$$

$$\therefore \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \ \ldots\ldots\ldots\ldots$$

Changing x into –x, we have

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots\ldots\ldots\ldots\ldots$$

$$\therefore log\frac{1+x}{1-x} = log(1+x) - log(1-x) = 2[\, x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots\ldots\ldots]$$

Q1. (b) Solution.

A function f(x , y) is said t be homogeneous of degree (or order) n in the variables x and y if it can be expressed in the form

$x^n \, \emptyset({}^{y}/_{x}) or\ y^n \emptyset({}^{x}/_{y})$

Composite functions - (i) if u = f(x, y) where x = ∅ (f) and y = x(t) then u is called a composite function of (the single variable) t and we can find $\frac{du}{dt}$ **.**

(ii) if x = f(x, y) where x = ∅(u, v) and y = x(u, v) then x is called a composite function of (two variables) u and v so that we can find $\frac{\partial z}{\partial u}$ and $\boldsymbol{\frac{\partial z}{\partial v}}$ them on Homogeneous function of degree n in x and y, then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} \ nu$$

$$u = x^n f({}^{y}/_{x}) \ ; \ \frac{\partial u}{\partial x} = n\, x^{n-1} f({}^{y}/_{x}) + x^n f'\,({}^{y}/_{x})(-{}^{y}/_{x^2})$$

$$\Rightarrow x\frac{\partial u}{\partial x} = nx^n f\left({}^{y}/_{x}\right) - x^{n-1}yf'\left({}^{y}/_{x}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Also $\frac{\partial u}{\partial y} = x^n f'\left({}^{y}/_{x}\right) * \frac{1}{x} = x^{n-1} f'\left({}^{y}/_{x}\right)$

$$\Rightarrow y\frac{\partial u}{\partial y} = x^{n-1}yf'\left({}^{y}/_{x}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Adding equation (1) and (2) we get

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nx^n f\left({}^{y}/_{x}\right) = nu$$

Q1.**(c) Solution.**

Here f(x, y) = $x^3 + y^3 - 3axy$

$f_x = 3x^2 - 3ay$, $f_y = 3y^2 - 3ax$, $r = f_{xx} = 6x$

$S = f_{xy} = -3a$, $t = f_{yy} = 6y$

Now

$f_x = 0$ & $f_y = 0$ and solving both we get two stationary points (0, 0) and (a , a)

Now

$rt - s^2 = 36\,xy - 9\,a^2$; At (0 , 0) , $rt - s^2 = -9a^2 < 0$

⇨ There is no extreme value at (0 , 0)

At (a, a), $rt - s^2 = 36\,a^2 - 9a^2 = 27\,a^2 > 0$

⇨ F(x, y) has extreme value at (a, a)

Now r = 6a; it a>0, r>0 so that f(x, y) has a minimum value at (a, a)

Min. value = $a^3 + a^3 - 3\,a^3 = -a^3$

It a<0, r<0 so that f(x, y) has a maximum value at (a ,a) ; max. value = $-a^3 - a^3 + 3a^3 = a^3$

Q1. (d) Solution.

The circum radius of a triangle ABC is given by

$$R = \frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C}$$

Now

a= 2R $\sin A$

Differentially are get da = 2R Cos A d_A or ${}^{d_a}/_{\cos A}$ = 2Rd_A

$\frac{d_b}{\cos B} = 2Rd_B$ & $\frac{d_c}{\cos C} = 2Rd_c$

Adding all three we get $\frac{d_a}{\cos A} + \frac{d_b}{\cos B} + \frac{d_c}{\cos C} = 2R\,(d_A + d_B + d_C$

= 0 (A+B+C = π,

OR

Q1 (e) Solution:

Equation of the curve is Y= c $\cosh(x/c)$ ....(1)

Diff. (1) w. r .to x, y1= c $\sinh(x/c) * \frac{1}{c}$ => y_1= $\sinh(x/c)$

Again diff. w. r. to x, we get $y_2 = \frac{1}{c}\cosh\frac{x}{c}$

$\therefore \varrho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{(1+\sinh^2 x/c)^{3/2}}{\frac{1}{c}\cosh\frac{x}{c}} = \frac{(\cosh^2)^{3/2}}{\frac{1}{c}\cosh\frac{x}{c}}$

$= c\cosh^2 x/c$ ....(2)

Now portion of the normal intercepted between the curve and the x-axis

= Length of Norma = $Y\sqrt{1+{y_1}^2}$

$= c\cosh\frac{x}{c}\sqrt{1+\sinh^2 x/c}$ = c $\cosh x/c \cosh x/c$ = c $\cosh^2 x/c$ ... (3)

From (2) & (3), we have ϱ = Length of normal

Again $\frac{\varrho}{(\text{ordinate})^2} = \frac{\varrho}{y^2} = \frac{c\cosh^2(x/c)}{c^2(\cosh^2(x/c)} = \frac{1}{c}$

∴ϱ varies as y^2, i.e., as square of the ordinate

Q2 (a) Solution

Gamma function if n is true, then

$\int_0^\infty e^{-x} x^{n-1}dx$, which is a function o x is called the Gamma function and it denoted by Γx.

Beta function – If m, n is positive, then the definite integral $\int_0^1 x^{m-1}(1-x)^{n-1}dx$, which is a function of m and n, is called the Beta function and is denoted by β(m, n).

$$\beta\ (m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx,\ m>0,\ n>0$$

Symmetry of Beta function

B (m, n) = $\int_0^1 x^{m-1}(1-x)^{n-1}$, $m > 0, n > 0$

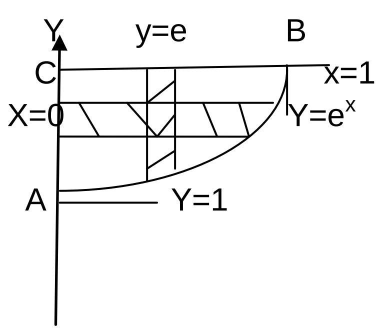
Now since

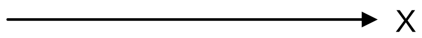
$$\int_0^a f(x)dx = \int_0^a f(a-x)dx$$

B (m, n) = $\int_0^1 (1-x)^{m-1}[1-(1-x)]^{n-1}dx = \int_0^1 x^{n-1}(1-x)^{m-1}dx = \beta\ (n, m)$

Q2 (b) Solution

Given limit shows that the region of integration is bounded by the curves





$Y = e^x$, $Y = e$

$X = 0$, $x = 1$

Hence $\int_0^1 \int_{e^x}^{e} \frac{dydx}{\log y} = \int_1^e \int_0^{\log y} \frac{dxdy}{\log y}$

$= \int_1^e \left(\frac{x}{\log y}\right)_0^{\log y} dy$

Q2 (c) Solution

$$\int_a^b sinx\ dy = \lim_{h\to\infty} h[\sin a + \sin(a+h) + \sin(a+2h) + \ \dots\dots\dots\dots + \sin(a + \overline{n-1}\, h$$

Where nh = b - a

$$= \lim_{h\to\infty} 2\left\{\frac{\frac{1}{2}h}{sin\frac{1}{2}h}\right\}.\sin\left(\frac{b-a}{2}\right).\sin\{a + \left(\frac{b-a-h}{2}\right)\}$$

$$= cos\ a - cos\ b$$

Q2 (d) Solution

Required volume

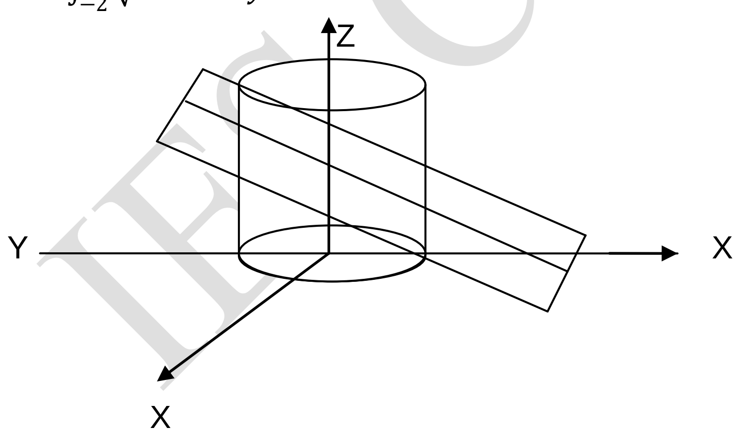
$= 2\int_{-2}^{2}\int_0^{\sqrt{4-Y^2}} Z\ dxdy$

$= 2\int_{-2}^{2}\int_0^{\sqrt{4-Y^2}} (4-Y)\ dxdy$

$= 2\int_{-2}^{2} (4-Y)[x]^{\sqrt{4-Y^2}} dy$

$= 2\int_{-2}^{2} (4-Y)\sqrt{4-Y^2 dy}$ $= = 2\int_{-2}^{2} 4\sqrt{4-Y^2 dy} - 2\int_{-2}^{2} Y\sqrt{4-Y^2 dy}$

$= 8\int_{-2}^{2}\sqrt{4-Y^2 dy}$ $=16\pi$



OR

Q2 (e) Solution

Let $P = \lim_{h\to\infty}[\left(1+\frac{1}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)\ldots\ldots\ldots(1+\frac{n^2}{n^2})^{1/4}$

Taking log on both sides are getting

$$\log p = \lim_{h\to\infty}\frac{1}{4}[\log\left(1+\frac{1^2}{n^2}\right)+\log\left(1+\frac{2^2}{n^2}\right)+\cdots\ldots\ldots..+\log\left(1+\frac{n^2}{n^2}\right)]$$

$= \lim_{h\to\infty}\frac{1}{4}\sum_{r=0}^{n}\log\left(1+\frac{r^2}{n^2}\right) = \int_0^1 1.\log(1+n^2)\,dx$

$\therefore \log p - \log 2 = \frac{\pi-4}{2}$

⇨ $log\frac{p}{2} = \frac{\pi-4}{2}$ => $p = 2e^{\frac{\pi-4}{2}}$

Unit 3

Q3 (a) Solution

The order of a diff. equation in the order of the highest order derivative accruing in the differential equation and the degree of a differential equation is the degree of the highest order derivative resent in the diff. equation.

e.g. $y = x\frac{dy}{dx} + (\frac{dy}{dx})^3$ is first order third degree diff. equation

Also the elimination of n arbitrary constants leads us to n^{th} order derivative and hence a diff equation of n^{th} order.

Q3 (b) Solution

$\frac{dy}{dx} = \frac{Y+\sqrt{x^2-y^2}}{x}$

$(homogeneous\ diff.equation)$

Put Y=vx so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$

V+xdv = v+ $\sqrt{1+v^2}$ => dv = dx

Integrating are get

$\log\{v+\sqrt{1+v^2}\} = \log x + \log c$

⇨ v+$\sqrt{1+v^2}$ = x.c

⇨ y + $\sqrt{x^2+y^2}$ = c x^2

Q3 (c) Solution

Equation is $\frac{dB}{dt} = kB$ where solution is B(t) = $C\,e^{kt}$

Let B_0 be the initial population at t=0 using the condition $B_0 = ce^0$

∴ c= B_0

Thus B = B_0e^{kt}

Since population triples i.e. becomes 3 B_0 between noon and 2 PM , i.e. in two hours, we use this condition to find k

$3B_0 = B_0e^{k\text{-}2}$

Thus k = $\frac{1}{2}\ln 3$ = 0.54930

To find the time at which the population become 100 times the original, i.e. , 100 $B_{0\,,}$ we put B =100 B_0 in the above equation and solve for t.

$100\ B_0 = B_0\ e^{0.3854930t}$

Solving t= $\frac{\ln 100}{0.54930}$ = 8.3837015

.i.e. , at 8.383 PM the population becomes the `100 times the original population.

Q3(d) Solution

Putting x = ^{z} so that z= logx and

Let D≡ $\frac{d}{dz}$

Then the given diff. equation reduces to

[D(D-1)(D-2)+3D(D-1)+D+1]y = e^z + z

⇨ (D^3+1) y =e^z + z ; A.e is m^3 +1 = 0

⇨ $(m+)(m^2\text{-}m+1)= 0$ => m=-1 , $\frac{1\pm\sqrt{3}\,i}{2}$

⇨ $\therefore$ C.F. = $C_1e^{-z}+e^{z/2}(C_2\cos\frac{\sqrt{3}}{2}z + C_3\sin\frac{\sqrt{3}}{2}z)$

⇨ P.I. $= \frac{1}{D^3+1}(e^z + z) = \frac{1}{(D^3+1)}e^z + \frac{1}{(D^3+1)}(z)$

⇨ $\frac{e^z}{z} + (1+D^3)^{-1}(z) = \frac{e^z}{z} + (1-D^3)(z)$ (Leaving higher order terms)

⇨ $\frac{e^z}{z} + z$

∴ The complete solution is

$Y = C_1e^{-z} + e^{z/2}(C_2\cos\frac{\sqrt{3}}{2}z + C_3\sin\frac{\sqrt{3}}{2}z) + \frac{e^z}{2} + \log x$

$\therefore Y = \frac{C_1}{x} + \sqrt{x}\,[c_2\cos\frac{\sqrt{3}}{2}(\log x) + C_3\sin\frac{\sqrt{3}}{2}(\log x)] + \frac{x}{2} + \log x$

Where C_1 and C_2 are the arb. Constant.

OR

Q 3 (e) Solution

$(D^2-2D+I)Y = e^x \tan x$

A.e. $m^2-2m+2 = 0$; $m = 1 \pm i$

$\therefore$ C.F. $= e^x[C_1 \cos x + C_2 \sin x]$

So $Y_1 = e^x \cos x$, $Y_2 = e^x \sin x$

$W = \begin{vmatrix} Y_1 & Y_2 \\ Y_1' & Y_2' \end{vmatrix} = e^{2x}$

P.I. $= -Y_1 \int \frac{Y_2 R}{W}\,dx + Y_2 \int \frac{Y_1 R}{W}\,dx$

$$= -e^x \cos x \int \frac{e^x \sin x . e^x \tan x}{e^{2x}}\,dx + e^x \sin x \int \frac{e^x \cos x . e^x \tan x}{e^{2x}}\,dx$$

$$= -e^x \cos x \int \frac{sin^2 x}{\cos x} + e^x \sin x \int \sin x\,dx$$

$$= -e^x \cos x \log(\sec x + \tan x)$$

$\therefore$ C.S. is $Y = e^x[C_1 \cos x + C_2 \sin x] - e^x \cos x \log(\sec x + \tan x)$

Unit 4

Q 4(a) Solution

Rank of $A \leq 3$ since A is of 3rd order

$|A|$ = 0 Now since $|A|$ = 0, rank of A<3

$i.e\,, r(A) \leq 2$

Consider the determinant of 2nd order sub matrices all second order determinants are equal to 0.

Since A is a non zero matrix. R(A) > 0

Thus the rank of A is one

Rank can be deduced by eliminatory transformation also

Q 4(b) Solution

AX = 0

$$\begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A \sim \begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & -8/7 \\ 0 & 0 & 0 \end{bmatrix}, \qquad i.e. \varrho(A) = z = r.$$

$But\ the\ number\ of\ variables = 3$

As n-r =3-2 = 1, the equation have infinite no. of solution

$$\begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or

x + 3y -2x = 0

$y - {}^{8}/_{7}\, z = 0$

Choose z= k and solving

we get x = $-\frac{10}{7}\, k,\ y = \frac{8}{7}\, k \,\&\, z = k$

Q 4(c) Solution

Let X be an eigen value of A. then

$AX = \lambda X \Rightarrow A^{-1}(AX) = A^{-1}(\lambda X) \Rightarrow (A^{-1}A)X = \lambda(A^{-1}X) \Rightarrow X = \lambda(A^{-1}X)$

$\Rightarrow \frac{1}{\lambda} X = A^{-1}X \Rightarrow A^{-1}X = \lambda^{-1}X$

λ^{-1} is an eigen value of A^{-1} and X is a corresponding eigen vector. Conversely suppose that k is an eigen value of A^{-1}

.since A is a non singular, A^{-1} is non singular

⇨ $\frac{1}{k}$ is an eigen value of A

⇨ *thus each eigen value of* A^{-1} *n equal to the reciprocal of some eigen value of A*

Q 4 (d) Solution

$|A - \lambda I| = 0$

$$\begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{bmatrix} = 0$$

Or (-2- ⅄)[-⅄(1- ⅄) – 12] -2 [-2⅄-6] -3[-4 +1(1=⅄)] = 0

Or $\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$ => ⅄ = -3,-3, 5

Corresponding to ⅄ = -3 the eigen vector are given by

(A+3I)X_1 = 0, or $\begin{bmatrix} -1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

We get only independent equation $x_1+x_2+x_3 = 0$

Let $x_3= k_1$, $x_2= k_2$ then $x_1= 3k_1 – 2k_2$

∴ the eigen vector are given by

$X_1 = \begin{bmatrix} 3k_1 - 2k_2 \\ k_2 \\ k_1 \end{bmatrix} = k_1\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + k_2\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$

Corresponding to $\lambda = 5,$ the eigen vector are given by

(A-5I) X_2 = 0 => $\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

⇨ $-7x_1+2x_2-3x_3 = 0$

⇨ $x_1-2x_2-3x_3 = 0$

⇨ $-x_1-2x_2-5x_3 = 0$

Or

4 (e) Solution

Char. Equation is $|A - \lambda I| = 0$ => $\begin{vmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix} = 0$; $\lambda = -1,5$

Let

$\lambda^n \equiv (\lambda^2 - 4\lambda - 5)Q(\lambda) + (a\lambda + b)$ ……….(1)

Where Q(λ) is quotient

Put λ= -1, $(-1)^n = -a + b$ ……………………..(2)

Put λ = 5, $5^n = 5a + b$ …………………………..(3)

Solving (2) and (3) we get

$A = \frac{5^n - (-1)^n}{6}$, $b = \frac{5^n - 5(-1)^n}{6}$

Replacing λ by matrix A in (1), we get

$A^n = (A^2 - 4A - 5I)\,Q(A) + (aA\_bI)$

$= 0 + aA + bI$ (by cayley Hamilton theorem then

$= aA + bI$

$= \{\frac{5^n - (-1)^n}{6}\}\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} + \{\frac{5^n + 5(-1)^n}{6}\}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Unit 5

Q 5 (a) Solution

Two propositional P(p,q, ….) and Q(p,q, ……)

Are said to be logically equivalent or simply equivalent if they have identical thruth tables

The truth table of the statement (pVq) ∧(~p∧~q)

p	q	pvq	~p	~q	(~p∧~q)	(pVq) ∧(~p∧~q)
T	T	T	F	F	F	F
T	F	T	F	T	F	F
F	T	T	T	F	F	F
F	F	F	T	T	T	F

statement has its value F for all, then it is Contradiction.

Q 5 (b) Solution

Let n vertices be v_1, v_2, v_3 ,……. v_n.

The no of edges drawn from v_1 to all other vertices are (n-1).

The no of edges drown from v_2 & all other vertices except v_1 are (n-2)

Similarly we can do for $v_3, v_4, v_5,$ ……………………v_n

Hence the total no of edges

= (n-1) + (n-2) + ……… 2 +1 = $\frac{1}{2}$ n(n-1)

Q5 (c) Solution

Representation of A is (z v y)∧(x v z) for B x v (a ∧ z)

So we have (x v (a ∧ z)) ∧ z

(A) And (B) are parallel. Hence the Boolean function for the whole circuit is

(x v y) ∧ (x v z) v (xv(x ∧ z)) ∧ z
= x ∧ (x v y) v y ∧ (x v z) v (x ∧ z)
= x v (y ∧ x) v(y ∧ z) v (x ∧ z)
= x v (x ∧ z) v (y ∧ z)
= x v (y ∧ z)

Q 5 (d) Solution

$$\sum_{i=0}^{k} n_i^2 \geq n^2 - (k-1)(2n-k)$$

We know that

$\sum_{i=0}^{k}(n_i - 1) = n - k$ Squaring both sides

We get $(\sum_{i=0}^{k}(n_i - 1))^2$ = n^2+k^2 -2nk

Or

$$\sum_{i=1}^{k}(n_i^2 - 2n_i) + k + non-negative\ cross\ term$$

= n^2+k^2 -2nk

Because $(n_i - 1) \geq 0\ for\ all\ i$

Therefore $\sum_{i=1}^{n} n_i^2 \leq n^2 + k^2 - 2nk - k + 2n$

= n^2 – (k-1)(2n-k)

Or

Q 5 (e) Solution

Ss f(x, y, z) = x.y' + x.z +x.y
= x.y' +x.z + x.y = x (y' + y) +x.z
= x + x.z = zx. (1+z) (∵ 1+z =1)
= x.1 = x
= x (y + y').(z + z')
= (x.y +x.y')(z +z')
= x.y.z +x.y.z' +x.y'.z+x.y'.z'